A COMBINATORIAL INTERPRETATION OF THE NUMBERS $6\,(2n)!\,/n!\,(n+2)!$

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ABSTRACT. It is well known that the numbers $(2m)!\,(2n)!/m!\,n!\,(m+n)!$ are integers, but in general there is no known combinatorial interpretation for them. When $m=0$ these numbers are the middle binomial coefficients $\binom{2n}{n}$, and when $m=1$ they are twice the Catalan numbers. In this paper, we give combinatorial interpretations for these numbers when $m=2$ or 3.

1. INTRODUCTION

The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

are well-known integers that arise in many combinatorial problems. Stanley [9, pp. 219–229], gives 66 combinatorial interpretations of these numbers.

In 1874 E. Catalan [1] observed that the numbers

$$\frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!} \tag{1.1}$$

are integers, and their number-theoretic properties were studied by several authors (see Dickson [2, pp. 265–266]). For $m=0$, (1.1) is the middle binomial coefficient $\binom{2n}{n}$, and for $m=1$ it is $2C_n$.

Except for $m=n=0$, these integers are even, and it is convenient for our purposes to divide them by 2, so we consider the numbers

$$T(m,n) = \frac{1}{2}\,\frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}. \tag{1.2}$$

Some properties of these numbers are given in [3], where they are called "super Catalan numbers". An intriguing problem is to find a combinatorial interpretation to the super Catalan numbers. The following identity [3, Equation (32)], together with the symmetry property $T(m,n)=T(n,m)$ and the initial value $T(0,0)=1$, shows that $T(m,n)$ is a positive integer for all m and n.

$$\sum_n 2^{p-2n}\binom{p}{2n} T(m,n) = T(m,m+p), \quad p \geq 0. \tag{1.3}$$

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Formula (1.3) allows us to construct recursively a set of cardinality $T(m,n)$ but we have not found any natural description of it for $m \geq 2$. Shapiro [8] gave a combinatorial interpretation to (1.3) in the case $m=1$, which is the Catalan number identity

$$\sum_n 2^{p-2n}\binom{p}{2n} C_n = C_{p+1}.$$

A similar interpretation works for the case $m=0$ of (1.3) (when multiplied by 2), which is the identity

$$\sum_n 2^{p-2n}\binom{p}{2n}\binom{2n}{n} = \binom{2p}{p}.$$

Another intriguing formula for $T(m,n)$, which does not appear in [3], is

$$1 + \sum_{m,n=1}^{\infty} C_m C_n x^m y^n = \left(1 - \sum_{m,n=1}^{\infty} T(m,n) x^m y^n\right)^{-1}. \tag{1.4}$$

Although (1.4) suggests a combinatorial interpretation for $T(m,n)$ based on a decomposition of pairs of objects counted by Catalan numbers, we have not found such an interpretation.

In this paper, we give a combinatorial interpretation for $T(2,n) = 6\,(2n)!\,/n!\,(n+2)!$ for $n \geq 1$ and for $T(3,n) = 60\,(2n)!\,/n!\,(n+3)!$ for $n \geq 2$. The first few values of $T(m,n)$ for $m=2$ and $m=3$ are as follows:

$m\backslash n$	0	1	2	3	4	5	6	7	8	9	10
2	3	2	3	6	14	36	99	286	858	2652	8398
3	10	5	6	10	20	45	110	286	780	2210	6460

We show that $T(2,n)$ counts pairs of Dyck paths of total length $2n$ with heights differing by at most 1. We give two proofs of this result, one combinatorial and one using generating functions. The combinatorial proof is based on the easily checked formula

$$T(2,n) = 4C_n - C_{n+1} \tag{1.5}$$

which we interpret by inclusion-exclusion.

Our interpretation for $T(3,n)$ is more complicated, and involves pairs of Dyck paths with height restrictions. Although we have the formula $T(3,n) = 16C_n - 8C_{n+1} + C_{n+2}$ analogous to (1.5), we have not found a combinatorial interpretation to this formula, and our proof uses generating functions.

Interpretations of the number $T(2,n)$ in terms of trees, related to each other, but not, apparently, to our interpretation, have been found by Schaeffer [7], and by Pippenger and Schleich [5, pp. 34].

2. The Main theorem

All paths in this paper have steps $(1,1)$ and $(1,-1)$, which we call *up steps* and *down steps.* A step from a point u to a point v is denoted by $u \to v$. The *level* of a point in a path is defined to be its y-coordinate. A *Dyck path* of *semilength* n (or of length $2n$) is a path that starts at $(0,0)$, ends at $(2n,0)$, and never goes below level 0. It is well-known that the number of Dyck paths of

semilength n equals the Catalan number C_n. The *height* of a path P, denoted by $h(P)$, is the highest level it reaches.

Every nonempty Dyck path R can be factored uniquely as $UPDQ$, where U is an up step, D is a down step, and P and Q are Dyck paths. Thus the map $P \mapsto (P, Q)$ is a bijection from nonempty Dyck paths to pairs of Dyck paths. Let $\mathbf{B}_n$ be the set of pairs of Dyck paths (P, Q) of total semilength n. This bijection gives $|\mathbf{B}_n| = C_{n+1}$, so by (1.5), we have $T(2, n) = 4C_n - |\mathbf{B}_n|$.

Our interpretatoin for $T(2, n)$ is a consequence of the following Lemma 2.1. We give two proofs of this lemma, one combinatorial and the other algebraic. The algebraic proof will be given in the next section.

Lemma 2.1. *For $n \geq 1$, C_n equals the number of pairs of Dyck paths (P, Q) of total semilength n, with P nonempty and $h(P) \leq h(Q) + 1$.*

Proof. Let $\mathbf{D}_n$ be the set of Dyck paths of semilength n, and let $\mathbf{E}_n$ be the set of pairs of Dyck paths (P, Q) of total semilength n, with P nonempty and $h(P) \leq h(Q) + 1$.

For a given pair (P, Q) in $\mathbf{E}_n$, since P is nonempty, the last step of P must be a down step, say, $u \to v$. By replacing $u \to v$ in P with an up step $u \to v'$, we get a path F_1. Now raising Q by two levels, we get a path F_2. Thus $F := F_1F_2$ is a path that ends at level 2 and never goes below level 0. The point v' belongs to both F_1 and F_2, but we treat it as a point only in F_2, even if F_2 is the empty path. The condition that $h(P) \leq h(Q) + 1$ yields $h(F_1) < h(F_2)$, which implies that the highest point of F must belong to F_2. See Figure 1 below.

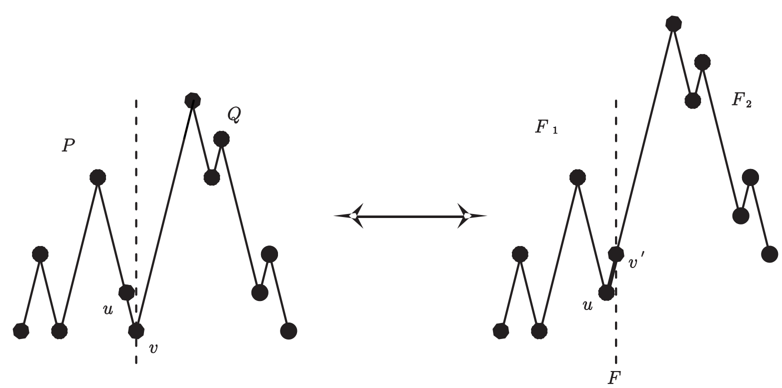

FIGURE 1. First step of the bijection

Now let y be the leftmost highest point of F (which is in F_2), and let $x \to y$ be the step in F leading to y. Then $x \to y$ is an up step. By replacing $x \to y$ with a down step $x \to y'$, and lowering the part of F_2 after y by two levels, we get a Dyck path $D \in \mathbf{D}_n$. See Figure 2 below.

With the following two key observations, it is easy to see that the above procedure gives a bijection from $\mathbf{E}_n$ to $\mathbf{D}_n$. First, x in the final Dyck path D is the rightmost highest point. Second, u in the intermediate path F is the rightmost point of level 1 in both F and F_1. $\square$

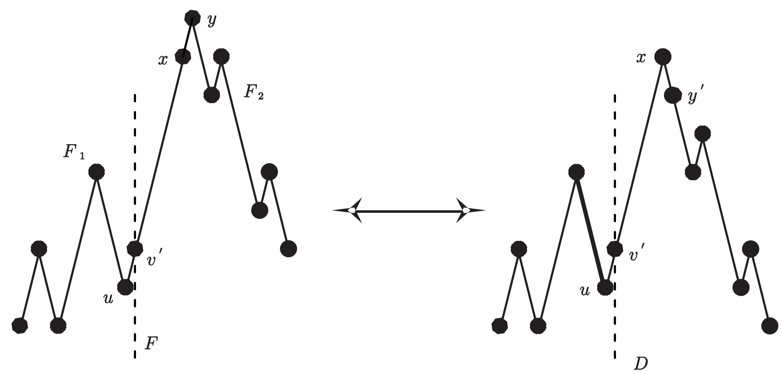

FIGURE 2. Second step of the bijection

Theorem 2.2. *For $n \geq 1$, the number $T(2,n)$ counts pairs of Dyck paths (P,Q) of total semilength n with $|h(P)-h(Q)| \leq 1$.*

Proof. Let F be the set of pairs of Dyck paths (P,Q) with $h(P) \leq h(Q)+1$, and let G be the set of pairs of Dyck paths (P,Q) with $h(Q) \leq h(P)+1$. By symmetry, we see that $|F|=|G|$. Now we claim that the cardinality of F is $2C_n$. This claim follows from Lemma 2.1 and the fact that if P is the empty path, then $h(P) \leq h(Q)+1$ for every $Q \in \mathbf{D}_n$.

Clearly we have that $F \cup G = \mathbf{B}_n$, and that $F \cap G$ is the set of pairs of Dyck paths (P,Q), with $|h(P)-h(Q)| \leq 1$. The theorem then follows from the following computation:

$$|F \cap G| = |F| + |G| - |F \cup G| = 4C_n - |\mathbf{B}_n| = 4C_n - C_{n+1}.$$

□

3. An Algebraic Proof and Further Results

In this section we give an algebraic proof of Lemma 2.1.

Let $c(x)$ be the generating function for the Catalan numbers, so that

$$c(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Then $c(x)$ satisfies the functional equation $c(x) = 1 + xc(x)^2$. Let $C = xc(x)^2 = c(x)-1$ and let G_k be the generating function for Dyck paths of height at most k. Although G_k is a rational function, a formula for G_k in terms of C will be of more use to us than the explicit formula for G_k.

Lemma 3.1. *For $k \geq -1$,*

$$G_k = (1+C)\frac{1-C^{k+1}}{1-C^{k+2}}. \tag{3.1}$$

Proof. Let P be a path of height at most $k \geq 1$. If P is nonempty then P can be factored as UP_1DP_2, where U is an up step, P_1 is a Dyck path of height at most $k-1$ (shifted up one unit), D is a down step, and P_2 is a Dyck path of height at most k. Thus $G_k = 1 + xG_{k-1}G_k$, so $G_k = 1/(1 - xG_{k-1})$. Equation (3.1) clearly holds for $k = -1$ and $k = 0$. Now suppose that for some $k \geq 1$,

$$G_{k-1} = (1+C)\frac{1-C^k}{1-C^{k+1}}.$$

Then the recurrence, together with the formula $x = C/(1+C)^2$, gives

$$\begin{aligned} G_k &= \left[1 - x(1+C)\frac{1-C^k}{1-C^{k+1}}\right]^{-1} = \left[1 - \frac{C}{1+C}\,\frac{1-C^k}{1-C^{k+1}}\right]^{-1} \\ &= \left[\frac{1-C^{k+2}}{(1+C)(1-C^{k+1})}\right]^{-1} = (1+C)\frac{1-C^{k+1}}{1-C^{k+2}}. \end{aligned}$$

□

We can prove Lemma 2.1 by showing that $\sum_{n=0}^{\infty} G_{n+1}(G_n - G_{n-1}) = 1 + 2C$; this is equivalent to the statement that the number of pairs (P,Q) of Dyck paths of semilength $n > 0$ with $h(P) \leq h(Q) + 1$ is $2C_n$.

Theorem 3.2.

$$\sum_{n=0}^{\infty}(G_n - G_{n-1})G_{n+1} = 1 + 2C.$$

Proof. Let

$$\Psi_k = \sum_{n=k}^{\infty} \frac{C^n}{1-C^n}.$$

Thus if $j < k$ then

$$\Psi_j = \sum_{n=j}^{k-1} \frac{C^n}{1-C^n} + \Psi_k. \tag{3.2}$$

We have

$$G_nG_{n+1} = (1+C)^2\frac{1-C^{n+1}}{1-C^{n+3}}$$

and

$$G_{n-1}G_{n+1} = (1+C)^2\frac{(1-C^n)(1-C^{n+2})}{(1-C^{n+1})(1-C^{n+3})}.$$

Let

$$S_1 = \sum_{n=0}^{\infty}\left(\frac{1-C^{n+1}}{1-C^{n+3}} - 1\right)$$

and

$$S_2 = \sum_{n=0}^{\infty}\left(\frac{(1-C^n)(1-C^{n+2})}{(1-C^{n+1})(1-C^{n+3})} - 1\right).$$

Then $\sum_{n=0}^{\infty}(G_n - G_{n-1})G_{n+1} = (1+C)^2(S_1 - S_2)$. We have

$$\frac{1-C^{n+1}}{1-C^{n+3}} - 1 = -\frac{(1-C^2)C^{n+1}}{1-C^{n+3}},$$

so $S_1 = -(1-C^2)C^{-2}\Psi_3$, and

$$\frac{(1-C^n)(1-C^{n+2})}{(1-C^{n+1})(1-C^{n+3})} - 1 = -\frac{(1-C)C^n}{(1+C)(1-C^{n+1})} - \frac{(1-C^3)C^{n+1}}{(1+C)(1-C^{n+3})},$$

so

$$S_2 = -\frac{1-C}{1+C}C^{-1}\Psi_1 - \frac{1-C^3}{1+C}C^{-2}\Psi_3.$$

Therefore

$$\begin{aligned} S_1 - S_2 &= \frac{1-C}{1+C}C^{-1}\Psi_1 + \left(\frac{1-C^3}{1+C} - (1-C^2)\right)C^{-2}\Psi_3 \\ &= \frac{1-C}{1+C}C^{-1}(\Psi_1 - \Psi_3) \\ &= \frac{1-C}{1+C}C^{-1}\left(\frac{C}{1-C} + \frac{C^2}{1-C^2}\right) = \frac{1+2C}{(1+C)^2}. \end{aligned}$$

Thus $(1+C)^2(S_1 - S_2) = 1 + 2C$. □

By similar reasoning, we could prove Theorem 2.2 directly: The generating function for pairs of paths with heights differing by at most 1 is

$$\sum_{n=0}^{\infty}(G_n - G_{n-1})(G_{n+1} - G_{n-2}),$$

where we take $G_{-1} = G_{-2} = 0$, and a calculation like that in the proof of Theorem 3.2 shows that this is equal to

$$\begin{aligned} 1 + 2C - C^2 &= 4c(x) - c(x)^2 - 2 = 4c(x) - \frac{c(x)-1}{x} - 2 \\ &= 1 + \sum_{n=1}^{\infty}(4C_n - C_{n+1})x^n = 1 + \sum_{n=1}^{\infty}T(2,n)x^n. \end{aligned}$$

Although the fact that the series in Theorem 3.2 telescopes may seem surprising, we shall see in Theorem 3.4 that it is a special case of a very general result on sums of generating functions for Dyck paths with restricted heights.

Lemma 3.3. *Let $R(z, C)$ be a rational function of z and C of the form*

$$\frac{zN(z,C)}{\prod_{i=1}^{m}(1 - zC^{a_i})},$$

where $N(z,C)$ is a polynomial in z of degree less than m, with coefficients that are rational functions of C, and the a_i are distinct positive integers. Let $L = -\lim_{z\to\infty} R(z,C)$. Then

$$\sum_{n=0}^{\infty} R(C^n, C) = Q(C) + L\Psi_1,$$

where $Q(C)$ is a rational function of C.

Proof. First we show that the lemma holds for $R(z,C) = z/(1-zC^a)$. In this case, $L = -\lim_{z\to\infty} R(z,C) = C^{-a}$ and

$$\sum_{n=0}^{\infty} R(C^n, C) = \sum_{n=0}^{\infty} \frac{C^n}{1-C^{n+a}} = C^{-a}\sum_{n=a}^{\infty} \frac{C^n}{1-C^n} = -\sum_{n=0}^{a-1} \frac{C^{n-a}}{1-C^n} + C^{-a}\Psi_1.$$

Now we consider the general case. Since $R(z,C)/z$ is a proper rational function of z, it has a partial fraction expansion

$$\frac{1}{z}R(z,C) = \sum_{i=1}^{m} \frac{U_i(C)}{1-zC^{a_i}}$$

for some rational functions $U_i(C)$, so

$$R(z,C) = \sum_{i=1}^{m} U_i(C)\frac{z}{1-zC^{a_i}}.$$

The general theorem then follows by applying the special case to each summand. □

Theorem 3.4. *If $i_1, i_1', \ldots, i_m$ are distinct integers, then*

$$\sum_{n=0}^{\infty} (G_{n+i_1} - G_{n+i_1'})G_{n+i_2}G_{n+i_3}\cdots G_{n+i_m}$$

is a rational function of C.

Proof. Apply Lemma 3.3 to

$$\sum_{n=0}^{\infty} (G_{n+i_1}G_{n+i_2}G_{n+i_3}\cdots G_{n+i_m} - 1)$$

and

$$\sum_{n=0}^{\infty} \left(G_{n+i_1'}G_{n+i_2}G_{n+i_3}\cdots G_{n+i_m} - 1\right).$$

□

4. A combinatorial interpretation for $T(3,n)$

It is natural to ask whether there are combinatorial interpretations to $T(m,n)$ for $m>2$ similar to Theorem 2.2. It is straightforward (with the help of a computer algebra system) to evaluate sums like (3.4) that count m-tuples of paths with height restrictions, and we find that sums like that in Theorem 3.4 involving products of m path generating functions may generally be expressed in the form

$$R_1(x)+R_2(x)\sum_{n=0}^{\infty} T(m,n)x^n,$$

where $R_1(x)$ and $R_2(x)$ are rational functions of x. However, for general m we have not found a set of m-tuples of paths counted by $T(m,n)$. We have found a set of paths counted by $T(3,n)$, though it is not as simple as one would like.

We need to consider paths that end at levels greater than 0. Let us define a *ballot path* to be a path that starts at level 0 and never goes below level 0.

In the previous section all our paths had an even number of steps, so it was natural to assign a path with n steps the weight $x^{n/2}$. We shall continue to weight paths in this way, even though some of our paths now have odd lengths.

Let $G_k^{(j)}$ be the generating function for ballot paths of height at most k that end at level j.

Lemma 4.1. *For $0\le j\le k+1$ we have*

$$G_k^{(j)} = C^{j/2}(1+C)\frac{1-C^{k-j+1}}{1-C^{k+2}} \tag{4.1}$$

Proof. The case $j=0$ is Lemma 3.1. Now let W be a ballot path counted by $G_k^{(j)}$, where $j>0$, so that W is of height at most k and W ends at level j. Then W can be factored uniquely as W_1UW_2, where W_1 is a path of height at most k that ends at level 0 and W_2 is a path from level 1 to level j that never goes above level k nor below level 1. Using $\sqrt{x}=\sqrt{C/(1+C)^2}=\sqrt{C}/(1+C)$, we obtain

$$G_k^{(j)} = G_k\sqrt{x}G_{k-1}^{(j-1)} = (1+C)\frac{1-C^{k+1}}{1-C^{k+2}}\cdot\frac{\sqrt{C}}{1+C}\cdot G_{k-1}^{(j-1)} = \sqrt{C}\frac{1-C^{k+1}}{1-C^{k+2}}G_{k-1}^{(j-1)},$$

and (4.1) follows by induction. □

We note an alternative formula that avoids half-integer powers of C,

$$G_k^{(j)} = x^{j/2}(1+C)^{j+1}\frac{1-C^{k-j+1}}{1-C^{k+2}},$$

which follow easily from (4.1) and the formula $\sqrt{x}=\sqrt{C}/(1+C)$.

Although we will not need it here, there is a similar formula for the generating function $G_k^{(i,j)}$ for paths of height at most k that start at level i, end at level j, and never go below level 0: for

$0 \le i \le j \le k+1$ we have

$$G_k^{(i,j)} = C^{(j-i)/2}(1+C)\frac{(1-C^{i+1})(1-C^{k-j+1})}{(1-C)(1-C^{k+2})}, \tag{4.2}$$

with $G_k^{(i,j)} = G_k^{(j,i)}$ for $i > j$.

We note two variants of (4.2), also valid for $0 \le i \le j \le k+1$:

$$\begin{aligned} G_k^{(i,j)} &= x^{(j-i)/2}(1+C)^{j-i+1}\frac{(1-C^{i+1})(1-C^{k-j+1})}{(1-C)(1-C^{k+2})}, \\ &= x^{-1/2}C^{(j-i+1)/2}\frac{(1-C^{i+1})(1-C^{k-j+1})}{(1-C)(1-C^{k+2})}. \end{aligned}$$

It is well known that $G_k^{(i,j)}$ is $x^{(j-i)/2}$ times a rational function of y, and it is useful to have an explicit formula for it as a quotient of polynomials. (See Sato and Cong [6] and Krattenthaler [4].) Let us define polynomials $p_n = p_n(x)$ by

$$p_n(x) = \sum_{0\le k\le n/2} (-1)^k \binom{n-k}{k} x^k.$$

The first few values are

$$\begin{aligned} p_0 &= 1 \\ p_1 &= 1 \\ p_2 &= 1-x \\ p_3 &= 1-2x \\ p_4 &= 1-3x+x^2 \\ p_5 &= 1-4x+3x^2 \\ p_6 &= 1-5x+6x^2-x^3 \end{aligned}$$

These polynomials can be expressed in terms of the Chebyshev polynomials of the second kind $U_n(x)$ by

$$p_n(x) = x^{n/2}U_n\left(\frac{1}{2\sqrt{x}}\right).$$

It is not difficult to show that

$$p_n = \frac{1-C^{n+1}}{(1-C)(1+C)^n},$$

and thus we obtain

$$G_k^{(i,j)} = x^{(j-i)/2}\frac{p_i p_{k-j}}{p_{k+1}},$$

for $0 \le i \le j \le k$, and in particular, $G_k^{(j)} = x^{j/2}p_{k-j}/p_{k+1}$ and $G_k = p_k/p_{k+1}$.

We can now describe our combinatorial interpretation of $T(3,n)$: $T(3,n)$ counts triples of ballot paths whose heights are k, $k-2$, and $k-4$ for some k, ending at levels 4, 3, and 2, together with

some additional paths of height at most 5. (Note that if a path of height $k-4$ ends at level 2, then k must be at least 6.) More precisely, let $H_k^{(j)}$ be the generating function for ballot paths of height k that end at level j, so that $H_k^{(j)} = G_k^{(j)} - G_{k-1}^{(j)}$. Then we have:

Theorem 4.2.

$$1 + \sum_{n=0}^{\infty} T(3, n+1)x^n = \sqrt{x} \sum_{k=6}^{\infty} H_k^{(4)} H_{k-2}^{(3)} H_{k-4}^{(2)} + 2G_1 + 2G_2 + G_3 + G_5. \tag{4.3}$$

Proof. First note that $T(m,n) = \frac{1}{2}(-1)^n 4^{m+n}\binom{m-\frac{1}{2}}{m+n}$, so

$$\begin{aligned}\tfrac{1}{2}(-1)^m(1-4x)^{m-1/2} &= \sum_{n=-m}^{\infty} \tfrac{1}{2}(-1)^n 4^{m+n}\binom{m-\frac{1}{2}}{m+n} x^{m+n} \\ &= \sum_{n=-m}^{0} \tfrac{1}{2}(-1)^n 4^{m+n}\binom{m-\frac{1}{2}}{m+n} x^{m+n} + \sum_{n=1}^{\infty} T(m,n)x^{m+n}.\end{aligned}$$

Setting $m = 3$, dividing both sides by x^4, and subtracting the first sum on the right from both sides gives

$$-\frac{(1-4x)^{5/2}}{2x^4} - \frac{10}{x} + \frac{15}{x^2} - \frac{5}{x^3} + \frac{1}{2x^4} = \sum_{n=0}^{\infty} T(3, n+1)x^n. \tag{4.4}$$

Using the method described in Lemma 3.3 and Theorem 3.4, we find, with the help of Maple, that the sum

$$\sqrt{x} \sum_{k=6}^{\infty} H_k^{(4)} H_{k-2}^{(3)} H_{k-4}^{(2)}$$

is equal to

$$-\frac{(1-4x)^{5/2}}{2x^4} - \frac{10}{x} + \frac{15}{x^2} - \frac{5}{x^3} + \frac{1}{2x^4} + 1 - \frac{2}{1-x} - 2\frac{1-x}{1-2x} - \frac{1-2x}{1-3x+x^2} - \frac{1-4x+3x^2}{1-5x+6x^2-x^3}.$$

Then (4.3) follows from (4.4), the formula $G_k = p_k/p_{k+1}$, and the formulas for p_k, $k = 1, \ldots, 6$. $\square$

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